B3 - Games, Graphs and Algebra : End-Semester Exam

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November 20, 2024. Time : 10.00 - 12.30 PM. **Maximum points : 40**

2 points will be deducted if you do not write your name on the answerscript.

There are two parts to the question paper - PART A and PART B. Read the instructions for each part carefully. Some simple notions and notations are recalled at the end.

1 PART A : TRUE-FALSE QUESTIONS - 10 Points.

Please write only TRUE or FALSE in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

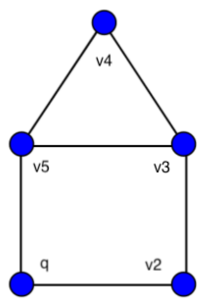
1. There exists an undirected graph G such that $\mathrm{Jac}(G) \cong \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3$.
2. For an undirected graph on n vertices with at most n edges, $r(D) \geq \deg(D)$.
3. $r(D) = 0$ where $D = \overrightarrow{1}$ is the all 1-vector.
4. For all undirected graphs G and any $q \in V$, we have that $\deg(v)[v - q] = \sum_{w \in V : w \sim v} [w - q]$.
5. If c is a superstable sandpile then $c_{max} - c$ is recurrent.
6. The maximal stable sandpile c_{max} is not recurrent on the complete graph.
7. If c is a stable sandpile in a sandpile graph, then so is $c + v$.
8. If D is a stable fixed energy sandpile, then so is $D + v$.
9. $P + Q$ is a stochastic matrix if P, Q are
10. PQ is a stochastic matrix if P, Q are.

2 PART B : 30 Points.

ALL QUESTIONS CARRY 10 POINTS. ATTEMPT ANY THREE OF THEM ONLY.

You are free to use any results that you have learnt in your course but please cite them clearly. Provide as many details as you can.

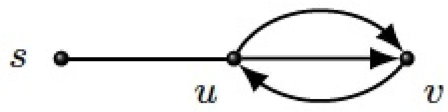
1. Consider the house graph as below. Compute the q-reduced divisors that are linearly equivalent to $D_1 = (-3, 2, 4, -2, 1)$, $D_2 = (2, 1, -5, 2, 2)$ and $D_3 = (0, -2, -2, 0, 0)$. Here we have represented divisor D as a vector $(D(q), D(v2), D(v3), D(v4), D(v5))$.



2. Determine the rank of all divisors on a tree and the cycle graph C_n for all $n \geq 1$.
3. Show that for a sandpile graph G, the zero configuration is recurrent iff G is s-acyclic.

4. Describe all the recurrents on the cycle graph C_n for all $n \geq 1$ (with an arbitrary sink). Compute the stationary density ζ_{st} for the fixed energy sandpile model on C_n. Here the cycle graph is considered as directed by orienting each edge in both directions.

5. A stable sandpile is recurrent iff it has no Forbidden subconfiguration. Prove or disprove this claim for the following two graphs - (i) A directed acyclic sandpile graph with a selfish vertex and (ii) Extended path graph as drawn below.



NOTIONS AND NOTATION

- All graphs G are finite, connected multi-graphs (undirected) or multi-digraphs (i.e., directed multi-graphs) depending on context.
- $r(D)$ denotes rank of a divisor $D = \sum_v D(v)v$. $\deg(D) = \sum_v D(v)$ denotes the degree of a divisor D.
- A sandpile graph G is (V, E, s) where (V, E) form a multi-digraph with a globally accessible sink s.
- A multidigraph is acyclic if there are no directed cycles.
- A sandpile graph G is s-acyclic if all its cycles involve s.
- $c_{max} := \sum_{v \in \tilde{V}} (\text{outdeg}(v) - 1)\, v$ is the maximal stable sandpile.
- A stable sandpile c is recurrent if for all $a \geq 0$, there exists $b \geq 0$ such that $(a + b)^{\circ} = c$.
- A stable sandpile is superstable if it has no legal script-firings.
- A fixed energy sandpile is a sandpile on a multi-digraph $G = (V, E)$ but without sinks.
- The stationary density $\zeta_{st} := \frac{1}{|B(G)|} \sum_{B \in B(G)} \frac{\deg(B)}{|V|}$ where $B(G)$ is the set of basic alive divisors of G w.r.t to an arbitrary sink.
- P is a stochastic matrix if all row-sums are 1.